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Finite Time Blowup for Semilinear Reactive–Diffusive Systems*

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1. INTRODUCTION

This paper deals with initial-boundary value problems for parabolic systems of the form

$$\begin{aligned}
&u_t - \Delta u = (1-v)\, f(u) && x \in \Omega \subset \mathbb{R}^n,\ t > 0\\
&v_t - \Delta v = (1-v)\, f(u), && \\
&u(x,0) = u_0(x) \geqslant 0, \quad 0 \leqslant v(x,0) = v_0(x) \leqslant 1, && x \in \Omega \\
&\frac{\partial u}{\partial n} + \mu u = 0, \quad \frac{\partial v}{\partial n} + \nu v = 0, && \mu, \nu \in [0, \infty]\ x \in \partial\Omega,\ t > 0,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $u_0(x)$ and $v_0(x)$ are continuous, bounded, nonnegative functions, and $f(s)$ is assumed to be C^2 with $f(s) > 0$, $f'(s) > 0$, $f''(s) > 0$, and $f(s) \gg s^2$ for $s \to \infty$. Additional assumptions will be imposed as needed in the next three sections.

Problem (1.1) arises as an approximating model for an exothermic chemical reaction taking place within a porous medium assuming one diffusing reactant and the Frank–Kamenetskii approximation $f(u) = e^u$ for the classical Arrhenius rate law. In this case u and v are chosen so that

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physically the scaled temperature of the reaction process above ambient is u and the scaled concentration is $1-v$.

Problem (1.1) [1, 2] has a unique nonnegative classical solution on $\pi_\sigma = \Omega \times [0, \sigma)$. By this, we mean a pair of nonnegative $C^{2,1}$ functions $(u(x, t), v(x, t))$ which satisfy (1.1) in π_σ. For a given solution (u, v) of (1.1), define

$$T = \sup\{\sigma > 0\colon (u, v) \text{ is a bounded solution of (1.1) on } \pi_\sigma\}. \tag{1.2}$$

The purpose of this paper is to determine in terms of u_0, v_0, μ, ν, f and Ω the cases $T = +\infty$ and $T < +\infty$. If $T = +\infty$, the solution exists for all time $t > 0$ and is global. If $T < +\infty$, then

$$\limsup_{t \to T} \left[\|u(t)\|_\infty + \|v(t)\|_\infty\right] = +\infty \tag{1.3}$$

since otherwise the solution could be extended beyond T. When (1.3) holds, with $T < \infty$, we say that the solution *blows up in finite time.*

The problem of blowup for scalar equations has been extensively discussed (see [1] for a comprehensive list of references). But for systems, the problem of blowup is much more delicate and relatively little is known [2, 6, 5, 7].

In [3], the second author together with Burnell and Wake investigated the steady-states (SS) for system (1.1). The results of that investigation can be summarized succintly by the diagram in Fig. 1.

In [2] we conjectured that if no steady-state solution exists for (1.1), then the solution of (1.1) blows up in finite time. We then proved for case (III_1), with $n = 1$, $f(u) = e^u$, and $\Omega = (0, 1)$ that finite-time blowup occurs.

v \ u	I D $\mu = \infty$	II R $0 < \mu < \infty$	III N $\mu = 0$
1 D $\nu = \infty$	I_1 D–D SS	II_1 R–D No SS for Ω large SS for Ω small	III_1 N–D No SS
2 R $0 < \nu < \infty$	I_2 D–R SS	II_2 R–R SS	III_2 N–R No SS
3 N $v = 0$	I_3 D–N SS	II_3 R–N SS	III_3 N–N SS

FIG. 1. D = Dirichlet, R = Robin, N = Neumann, SS = steady state.

The outline of the present paper is as follows. In Section 2, we prove finite-time blowup for case (III_1) with $\Omega = B_1$, u_0, v_0 radially symmetric, and f as general as possible within the stated class. In Section 3 we again prove finite time blowup for (III_1) for Ω a convex domain with a uniform interior sphere condition and $f(u) = e^u$. In Section 4 we prove blowup for (II_1), provided $\Omega = B_R$ is a ball with sufficiently large radius R so that no steady state solutions of (1.1) exist and $f(u) = e^u$. In the final section we complete our discussion of Fig. 1.

In each of the next three sections, we will prove the following theorem

THEOREM 1.1. *The solution* $(u(x, t), v(x, t))$ *of the initial-boundary value problem* (1.1) *blows up in finite time* $T < \infty$ *in the sense of* (1.3).

We give three separate proofs of this theorem for in each case the assumptions or the actual problem considered differ.

2. NEUMANN–DIRICHLET BOUNDARY CONDITIONS, I

Let $\Omega = B_1$, a ball of radius 1 in $\mathbb{R}^n$, and consider

$$\begin{aligned} & u_t - \Delta u = (1 - v) f(u), && \Omega = B_1 \\ & v_t - \Delta v = (1 - v) f(u) && \\ & u(x, 0) = u_0(x) \geqslant 0, \qquad 0 \leqslant v(x, 0) = v_0(x) \leqslant 1, && x \in B \\ & \frac{\partial u}{\partial n}(x, t) = 0, \qquad v(x, t) = 0, && x \in \partial B,\ t > 0 \end{aligned} \tag{2.1}$$

with the additional assumptions: $u_0(x)$ is radially symmetric and radially increasing, $v_0(t)$ is radially symmetric and radially decreasing,

$$\Delta u_0(x) + (1 - v_0(x)) f(u_0(x)) \geqslant 0 \qquad \text{and} \qquad \Delta(u_0(x) - v_0(x)) \geqslant 0.$$

We now proceed to prove Theorem 1.1 for IBVP (2.1) (a special case of (1.1)). We begin by considering an equivalent problem. Set $h = u - v$, then

$$\begin{aligned} & u_t - \Delta u = (1 + h - u) f(u) && \\ & h_t - \Delta h = 0 && \\ & u(x, 0) = u_0(x), \qquad h(x, 0) = u_0(x) - v_0(x), && x \in B \\ & \frac{\partial u}{\partial n(x)}(x, t) = 0, \qquad h(x, t) = u(x, t), && x \in \partial B,\ t > 0. \end{aligned} \tag{2.2}$$

By the assumptions on $u_0(x)$ and $v_0(x)$, we have

LEMMA 2.1. *u and h are radially symmetric with*

$$u_t \geqslant 0, \qquad h_t \geqslant 0 \qquad \text{on } \pi_T.$$

COROLLARY. *If* $T=\infty$, *then* $\sup_{x \in B} u(x, t) \to \infty$ *as* $t \to \infty$.

Proof. From the lemma, if u were bounded, then there would be some $u^*(x)$, $h^*(x)$ such that $u \to u^*$, $h \to h^*$ as $t \to \infty$. u^*, $u^* - h^*$ would necessarily be a steady-state to (2.1), contradicting the nonexistence of such solutions.

Using the equivalent formulation (2.1), we now observe

LEMMA 2.2. $u_r \geqslant 0$ *for* $r \in [0, 1]$, $t > 0$.

Proof. If $h(r, t)$ is the solution of $0 \leqslant h_t = h_{rr} + ((n-1)/r)\, h_r \leqslant (1/r^n)(r^n h_r)_r$, then $h_r(r, 0) \geqslant 0$ and $h_t(r, t) \geqslant 0$ for $0 \leqslant r \leqslant 1$, $t \geqslant 0$. Since $(r^n h_r)$ is an increasing function of r and vanishes for $r = 0$, $h_r(r, t) \geqslant 0$ for $0 \leqslant r \leqslant R$.

From (2.2), $u(r, t)$ satisfies $u_t - \Delta u = (1 + h - u)\, f(u)$. Differentiating with respect to r, we have

$$u_{rt} - u_{rrr} - \frac{n-1}{r} u_{rr} + \frac{n-1}{r^2} u_r$$
$$+ (f - (1 + h - u)\, f'(u))\, u_r = f h_r \geqslant 0$$

with $u_r(1, t) = 0$, $u_r(r, 0) \geqslant 0$, and $u_r(0, t) = 0$. This gives $u_r(r, t) \geqslant 0$ for $0 \leqslant r \leqslant 1$, $0 \leqslant t < T$, or there exists some $t_0 \in [0, T)$ and some $s(t)$ defined for $t_0 \leqslant t < t_1 \leqslant T$ with $0 < s(t) < 1$ for $t_0 < t < t_1$, $s(t_0) = 0$ and such that $u_r(r, t) \geqslant 0$ for $0 \leqslant r \leqslant 1$, $0 \leqslant t \leqslant t_0$, $u_r(r, t) \geqslant 0$ for $s(t) \leqslant r \leqslant 1$, $t_0 \leqslant t < t_1$, $u_r(s(t), t) = 0$ for $t_0 \leqslant t < t_1$, and for $t_0 < t < t_1$ there is some $r < s(t)$ with $u(r, t) > u(s, t)$.

Assume such t_0 and s exist, we define $U(r, t)$ for $0 \leqslant t < t_1$ by

$$U(r, t) = \begin{cases} u(r, t) & \text{for } \ 0 \leqslant r \leqslant 1,\ 0 \leqslant t \leqslant t_0, \text{ and} \\ & \qquad s(t) \leqslant r \leqslant 1,\ t_0 \leqslant t < t_1 \\ u(s(t), t) & \text{for } \ 0 \leqslant r \leqslant s(t),\ t_0 \leqslant t < t_1. \end{cases}$$

Then $U(r, 0) = u(r, 0)$, $U(1, t) = u(1, t)$, U and U_r are continuous. For $0 < r < 1$, $0 < t < t_0$, and for $s(t) < r < 1$, $t > t_0$,

$$U_t - \Delta U - (1 + h - U)\, f(U) = 0.$$

For $0 \leqslant r < s(t)$, $t_0 < t$,

$$
\begin{aligned}
& U_t - \Delta U - (1+h-U) f(U) \\
& \quad = u_t(s(t), t) - (1 - h(r, t) - u(s, t)) f(u(s, t)) \\
& \qquad \text{since} \quad u_r(s, t) = 0 \\
& \quad = u_{rr}(s(t), t) + (h(s, t) - h(r, t)) f(u(s, t)) \\
& \qquad \text{again since} \quad u_r(s, t) = 0 \\
& \quad \geqslant (h(s(t), t) - h(r, t)) f(u(s, t)) \\
& \qquad \text{since } u_r \geqslant 0 \text{ for } r > s,\ u_r = 0 \text{ for } r = s \text{ gives } u_{rr}(s, t) \geqslant 0 \\
& \quad \geqslant 0 \qquad \text{since} \quad h_r \geqslant 0.
\end{aligned}
$$

This implies U is an upper solution for (2.2_1) so $u(r, t) \leqslant U(r, t) = u(s(t), t)$ for $0 \leqslant r \leqslant s(t)$. This contradicts the assumptions on t_0 and s. Thus, $u(r, t)$ is increasing in r and $u_r \geqslant 0$ for $0 \leqslant r \leqslant 1$, $0 \leqslant t \leqslant T$.

Define

$$M(t) \equiv u(1, t).$$

Then

Lemma 2.3. *$M(t)$ is unbounded as $t \to T \leqslant \infty$.*

Proof. If $T = \infty$, result follows by the unboundedness of u and h. For $T < \infty$, it is immediate.

Lemma 2.4. *$u(r, t) \leqslant M(t)$ for $r \in (0, 1)$, $0 < t < T$.*

Proof. This is immediate from the definition of M and Lemma 2.2.

Theorem 2.1. *If $w(x)$ is the solution of*

$$
\begin{cases} \Delta w + (1 - w) f(\bar{M}) = 0, & B \\ w = 0, & \partial B, \end{cases} \tag{2.3}
$$

where $\bar{M} = \sup_{[0,\tau)} M(t)$, then $v \leqslant w$ for $0 < t \leqslant \tau$, $x \in B$, where v is the solution of

$$
\begin{aligned}
& v_t = \Delta v + (1 - v) f(u) \\
& v(x, 0) = v_0, \qquad x \in B \\
& v(x, t) = 0, \qquad x \in \partial B,\ 0 < t \leqslant \tau
\end{aligned}
$$

provided that τ is sufficiently close to T.

Proof. We have $0 \leqslant v \leqslant 1$. By taking τ sufficiently close to T, we may assume $\bar{M}$ is sufficiently large that $1 \geqslant w(x) \geqslant v_0(x)$ on B, where $w(x)$ is solution of (2.3). Since

$$\begin{aligned} 0 &= w_t - \Delta w - (1-w)\, f(\bar{M}) \\ &\leqslant w_t - \Delta w - (1-w)\, f(u), \end{aligned}$$

$w(x)$ is an upper solution of (2.1) and hence

$$v(x, t) \leqslant w(x) \qquad \text{for} \quad x \in B,\ 0 \leqslant t \leqslant \tau.$$

COROLLARY. *$v(x, t) \leqslant w(x)$ for all t close to T, where w satisfies* (2.3) *with $M(t)$ replacing $\bar{M}(\tau)$.*

Proof. Take $t = \tau$ and note $M(t)$ is increasing so $\bar{M}(t) = M(\tau)$.

THEOREM 2.2. *$v(r, t) \leqslant -w_r(1)(1-r) \leqslant [f(M(t))]^{1/2}(1-r)$ for t sufficiently close to T.*

Proof. Observe that

$$v(r, t) \leqslant w(r) \leqslant -w_r(1)(1-r)$$

for $t \leqslant T$ if $w_{rr}(r) \leqslant 0$.

To prove $w_{rr} \leqslant 0$, we note $w_{rr}(0) < 0$ and $w_r(r) < 0$ for $r > 0$. Then differentiating (2.3)

$$\begin{aligned} w_{rrr} &= -\frac{n-1}{r} w_{rr} + \frac{n-1}{r^2} w_r + w_r f(M) \\ &< -\frac{n-1}{r} w_{rr}. \end{aligned}$$

This would give $w_{rrr} < 0$ if $w_{rr} = 0$ for some r. By taking the smallest such r we have a contradiction and the assertion $w_{rr} \leqslant 0$ follows.

Set

$$W(r) = 1 - \frac{\cosh r f^{1/2}}{\cosh f^{1/2}},$$

then

$$W(1) = 0, \qquad W'(0) = 0$$

and

$$W'' + \frac{n-1}{r} W' + (1-W)\, f \leqslant 0.$$

This implies $W(r)$ is an upper solution of (2.3) and hence $W(r) \geqslant w(r)$. From this,

$$w'(1) \geqslant W'(1) = -f^{1/2} \tanh f^{1/2} \geqslant -f^{1/2}.$$

Thus,

$$v(r, t) \leqslant [f(M(t))]^{1/2} (1-r)$$

for t sufficiently close to T.

COROLLARY. $1-v(r, t) \geqslant 1-\sqrt{f(M(t))}\,(1-r)$ *for* $1-(1/f(M)^{1/2}) \leqslant r \leqslant 1$ *and* t *sufficiently near* T.

THEOREM 2.3. $u(r, t) \geqslant M(t)-1$ *for* $1-r \leqslant (1/\sqrt{f(M(t))}$ *for* $M(t)$ *large enough, i.e.,* t *sufficiently close to* T.

Proof. Since

$$u_{rr} + \frac{n-1}{r} u_r = u_t - (1-v) f(u) \geqslant -(1-v) f(u) \geqslant -f(M),$$

multiplying both sides by r^{n-1} and integrating from r to 1,

$$\int_r^1 (r^{n-1} u_r)_r \, dr \geqslant -f(M) \int_r^1 r^{n-1} \, dr$$

gives

$$-r^{n-1} u_r \geqslant -\frac{f(M)}{n} (1-r^n).$$

Then

$$u_r \leqslant \frac{f(M)}{n} (r^{1-n} - r).$$

Integrating

$$\int_r^1 u_r \, dr \leqslant \frac{f(M)}{n} \int_r^1 (r^{-(n-1)} - r) \, dr$$

$$M - u(r, t) \leqslant \frac{f(M)}{n} \int_{1-(1/\sqrt{f(M)})}^1 (r^{-(n-1)} - r) \, dr$$

for

$$
\begin{aligned}
r &\geqslant 1-\frac{1}{\sqrt{f(M)}} \\
&= \frac{f(M)}{n}\int_0^{f^{-1/2}} \left[(1-y)^{-(n-1)}-(1-y)\right] dy \\
&\leqslant \frac{f(M)}{n}\int_0^{f^{-1/2}} 2ny\, dy \qquad \text{for } f \text{ sufficiently large} \\
&= 1. \quad \blacksquare
\end{aligned}
$$

Set $U(t)=\int_{B_1} u(x, t)\, dx$, where u is the solution of

$$
\begin{aligned}
&u_t = \Delta u + (1-v) f(u) \\
&u(x, 0) = u_0(x) \\
&\frac{\partial u}{\partial n}(x, t) = 0.
\end{aligned}
$$

Under the assumption that $u(x, t)$ exists globally, then $U(t) \to \infty$ as $t \to \infty$. This follows from $M \to \infty$ as $t \to \infty$ and $u_t - \Delta u \geqslant 0$ with $u = M$ on ∂B.

Then

$$
\begin{aligned}
U'(t) &= \int_B u_t\, dx = \int_B (\Delta u + (1-v) f(u))\, dx \\
&= \int_B (1-v) f(u)\, dx \qquad \text{since} \quad u_n = 0 \\
&\geqslant \omega_n \int_{1-f(M)^{-1/2}}^{1} r^{n-1}(1-v) f(u)\, dr \\
&\geqslant \omega_n f(M-1) \int_{1-f(M)^{-1/2}}^{1} r^{n-1}\left[1-f(M)^{1/2}(1-r)\right] dr
\end{aligned}
$$

by corollary to Theorem 2.2 and 2.3

$$
\begin{aligned}
&\geqslant \omega_n f(M-1)\, f^{-1/2}(M) \int_0^1 (1-y)(1-f^{-1/2} y)^{n-1}\, dy \\
&\geqslant \omega_n f(M-1) \frac{f^{-1/2}(M)}{2} \int_0^1 (1-y)\, dy
\end{aligned}
$$

for $f(M)$ large enough

$$= \frac{\omega_n f(M-1)}{4f^{1/2}(M)}.$$

Thus,

$$U'(t) \geqslant \frac{\omega_n}{4} \frac{f(M-1)}{f^{1/2}(M)}$$

and

$$U(t) = \int_{B_1} u\,dx \leqslant M \frac{\omega_n}{n}.$$

If $(f \circ d)/f^{1/2}$ is increasing at least for sufficiently large argument, where $d(s) = s-1$, e.g., $f(s) = A(s+B)^p$ or some faster growing function, then

$$U' \geqslant \frac{\omega_n}{4} \frac{f \circ d(M)}{f^{1/2}(M)} \geqslant \frac{\omega_n}{4} \frac{f \circ d(nU/\omega_n)}{f^{1/2}(nU/\omega_n)}$$

or

$$U' \geqslant \frac{\omega_n}{4} \frac{f(nU/\omega_n - 1)}{f^{1/2}(nU/\omega_n)}.$$

EXAMPLE. If $f(u) = u^p$, $p > 1$, then

$$\frac{f(u-1)}{f(u)^{1/2}} = \frac{(u-1)^p}{u^{p/2}} = \frac{u^p}{u^{p/2}}(1-u^{-1}) \geqslant cu^{p/2}$$

for u large. Thus, $U' \geqslant cU^{p/2}$ for $t > t_0$ for some t_0 with $U(t_0) = \int_{B_1} u(x, t_0)\,dx = U_0 > 0$. If $p > 2$, $U(t)$ blows up at $t_B \leqslant t_0 + U_0^{1-p/2}/(p/2-1)$.

We conclude that $(u(x,t), v(x,t))$ blows up at some $T \leqslant t_B$.

Likewise, we obtain blowup whenever f, f', $f'' > 0$, $(f \circ d(s))/f^{1/2}(s)$ increases for s large, and $f(s)$ grows faster than s^2 for s large.

Remarks. (1) We note that by making a comparison in the u, h formulation with some u^*, v^* which have initial data u_0^*, v_0^* both constant and satisfy $u_0^* \leqslant u_0$ and $0 \leqslant u_0^* - v_0^* \leqslant u_0 - v_0$, we then obtain blowup for more general initial data (including asymmetric data) satisfying $u_0 \geqslant v_0$.

(2) By starting from $t = \varepsilon > 0$, we can drop the consistency and smoothness requirements on v_0.

(3) The proofs stand for arbitrary balls B_R.

3. Neumann–Dirichlet Boundary Conditions, II

Let Ω be a convex domain in $\mathbb{R}^n$ satisfying an interior sphere condition of the following kind. For each $x_0 \in \partial\Omega$ there exists a ball $B = B_{x_0} = B_R(x_1) \subset \Omega$ centered at x_1 of radius $R = R(x_0)$ with $x_0 \in \partial B$ such that the supporting hyperplane H to Ω through x_0 has normal parallel to $x_0 - x_1$ and $\inf_{x_0 \in \partial\Omega} R(x_0) = r \geqslant 1$. We note that this assumption $r \geqslant 1$ simplifies the proof of finite time blowup. With a slight modification of the proof we may assume $r \in (0, 1)$.

Consider on such a domain Ω

$$\begin{aligned}
&u_t - \Delta u = (1 - v)\, e^u \\
&v_t - \Delta v = (1 - v)\, e^u \\
&u(x, 0) = u_0(x) \geqslant 0, \qquad 0 \leqslant v(x, 0) = v_0(x) \leqslant 1, \qquad x \in \Omega \\
&\frac{\partial u}{\partial n(x)}(x, t) = 0, \qquad v(x, t) = 0, \qquad x \in \partial\Omega.
\end{aligned} \tag{3.1}$$

We now proceed to prove Theorem 1.1 for IBVP (3.1).

Set $h = u - v$, then (3.1) is equivalent to

$$\begin{aligned}
&v_t - \Delta v = (1 - v)\, e^{v + h} \\
&h_t - \Delta h = 0 \\
&v(x, 0) = v_0(x), \qquad h(x, 0) = u_0(x) - v_0(x), \qquad x \in \Omega \\
&(v + h)_{n(x)}(x, t) = 0, \qquad v(x, t) = 0, \qquad h(x, t) = u(x, t), \qquad x \in \partial\Omega,\ t > 0.
\end{aligned} \tag{3.2}$$

For any $x_0 \in \partial\Omega$, let $B = B_{x_0} = B_1(x_1)$ be the interior ball. Define for each $t \in [0, T)$

$$\underline{h}^*(t) \equiv \inf_{x_0 \in \partial\Omega} \inf_{x \in B_{x_0}} h(x, t) \tag{3.3}$$

and then consider

$$\begin{aligned}
&v_t - \Delta v = (1 - v)\, e^{v + \underline{h}^*(t)} \\
&v(x, 0) = 0 \leqslant v_0(x), \qquad x \in B \\
&v(x, t) = 0, \qquad x \in \partial B,\ t > 0.
\end{aligned} \tag{3.4}$$

Let $v^*(x, t)$ be the solution of (3.4), then

$$v^*(x, t) \leqslant v(x, t) \qquad \text{for} \quad x \in B$$

as long as both exist.

We obtain a second comparison solution by letting $w(x, t)$ be the solution of

$$\begin{aligned} &w_t = \Delta w + (1 - w), && x \in B,\ t > 0 \\ &w(x, 0) = 0 \leqslant v_0(x), && x \in B \\ &w(x, t) = 0, && x \in \partial B,\ t > 0. \end{aligned} \tag{3.5}$$

Then

$$w(x, t) \leqslant v^*(x, t) \qquad \text{on } B \tag{3.6}$$

as long as $v^*(x, t)$ exists.

We also can obtain a useful lower bound for the outer normal directional derivative of w.

THEOREM 3.1. *For some $c > 0$ there exists $t_c > 0$ such that*

$$-\frac{\partial w}{\partial n(x)}(x, t) > c \qquad \textit{for} \quad x \in B,\ t > t_c \tag{3.7}$$

Proof. Note that $\alpha(x, t) \equiv 0$ is a lower solution for (3.5) and that the steady-state solution $\beta(x)$ of

$$\begin{aligned} -\Delta u &= 1 - u, && x \in B \\ u(x) &= 0, && x \in \partial B \end{aligned}$$

is an upper solution of (3.5). Hence $0 \leqslant w(x, t) \leqslant \beta(x)$ with $-\partial\beta(x)/\partial n(x) > 0$. Since $w_t \geqslant 0$, $w(x, t) \to \beta(x)$ as $t \to \infty$ uniformly on B (since [2]). Thus, for any $c \in (0, -\partial\beta(x)/\partial n(x))$, there exists $t_c > 0$ such that (3.7) holds.

COROLLARY.

$$-\frac{\partial v^*(x, t)}{\partial n(x)} > c > 0 \qquad \textit{for} \quad x \in \partial B,\ t > t_c. \tag{3.8}$$

Proof. This follows immediately from (3.6) and (3.7).

Let $H = H_{x_0}$ be the supporting halfspace of Ω at x_0. Then $B \subset \Omega \subset H$. On this supporting halfspace, let $h^*(x, t)$ be the solution of

$$\begin{aligned} &h_t - \Delta h = 0, && x \in H \\ &h(x, 0) = \inf_{\Omega} [u_0(x) - v_0(x)], && x \in H \\ &h_{n(x)}|_{x \in \partial H} = -v^*_{n(x)}|_{x \in \partial B}, && t > 0. \end{aligned} \tag{3.9}$$

Then using [4, p. 475, Theorem 2] we have that $h^*(x, t) \leqslant h(x, t)$ for all $x \in \Omega$ and hence

$$h^*(x, t) \leqslant h(x, t) \qquad \text{for all} \quad x \in B \tag{3.10}$$

as long as both solutions exist.

Assume $T = \infty$, i.e., the solution (u, v) of (3.1) exists globally on Ω. We will show that this assumption leads to a contradiction. Since $h^*(x, t)$ is the solution of (3.9), by (3.8) we have that $h^*(x, t)$ is unbounded on B as $t \to \infty$. Thus, $h \geqslant h^*$ and $\underline{h}^*(t)$, defined by (3.3), tend to ∞ as $t \to T = \infty$.

For each natural number N, define

$$T_N \equiv \inf\{t > 0 \colon h(x, s) \geqslant N \text{ for } x \in \Omega,\ s > t\}. \tag{3.11}$$

Then obviously $0 < T_N < T_{N+1}$, $h(x, T_N) \geqslant N$ for all $x \in \Omega$, and

$$T_1 + \sum_{N=1}^{\infty} (T_{N+1} - T_N) = T.$$

We now proceed to construct a sequence $\{\tau_N\}$ with $\tau_N > T_{N+1} - T_N$ such that $\sum_{N=1}^{\infty} \tau_N < \infty$. This will contradict the assumption $T = \infty$.

For each $N \in \mathbb{N}$, let (v_N, h_N) be the unique solution of

$$\begin{aligned}
&v_t - \Delta v = (1 - v)\, e^{v + h + N}, && \Omega\\
&h_t - \Delta h = 0, && \Omega\\
&v(x, 0) = 0, \qquad h(x, 0) = 0, && \Omega\\
&(v + h)_{n(x)} = 0, \qquad v = 0, && \partial\Omega.
\end{aligned} \tag{3.12_N}$$

THEOREM 3.2. $h_N(x, t) \leqslant h_N(x, t) + N \leqslant h(x, t + T_N)$, $x \in \Omega$, $t > 0$.

Proof. The proof follows using the h, $u = v + h$ formulation of (3.12_N) to obtain a straightforward comparison via the maximum principle.

Set

$$t_N = \inf\{t > 0 \colon h_N(x, s) \geqslant 1,\ x \in \Omega,\ s \geqslant t\},$$

then $h_N(x, t_N) \geqslant 1$ for all $x \in \Omega$ and by Theorem 3.2

$$T_{N+1} \leqslant T_N + t_N.$$

Let $v_N^*(x, t)$ be the solution of

$$\begin{aligned} &v_t - \Delta_x v = (1-v)\, e^{h^*(t)+v+N}, && x \in B_1,\ t>0 \\ &v(x,0)=0, && x\in B_1 \\ &v(x,t)=0, && x \in B_1,\ t>0 \end{aligned} \tag{3.13_N}$$

and let $w_N(x, t)$ be the solution of

$$\begin{aligned} &w_t - \Delta_x w = (1-w)\, e^{N}, && x \in B_{e^{-N/2}},\ t>0 \\ &w(x,0)=0, && x\in B_{e^{-N/2}} \\ &w(x,t)=0, && x \in B_{e^{-N/2}},\ t>0 \end{aligned} \tag{3.14_N}$$

then on $B_{e^{-N/2}}$, $w_N(x, t) \leqslant v_N(x, t)$.

Rescale by letting

$$x = e^{-N/2} y, \qquad t = e^{-N}\tau. \tag{3.15}$$

Then $w_N(y, \tau)$ is a solution of

$$\begin{aligned} &w_\tau - \Delta_y w = 1 - w, && y \in B_1,\ \tau>0 \\ &w(y,0)=0, && y\in B_1 \\ &w(y,\tau)=0, && y \in \partial B_1,\ \tau>0. \end{aligned} \tag{3.16}$$

We then have, completely analogous to Theorem 3.1,

THEOREM 3.3. *For some $c \in (0, 1)$, there exists $\tau_c > 0$ such that*

$$-\frac{\partial w_N}{\partial n(y)}(y, \tau) > 0 \qquad \textit{for} \quad y \in \partial B_1,\ \tau > \tau_c.$$

Hence,

COROLLARY. $-(\partial v_N/\partial n(x))(x, t) \geqslant -\partial v_N^*(x, t)/\partial n(x) > ce^{N/2}$ *for* $x \in \partial B_1$, $t > \tau_c e^{-N}$.

Let $h_N^*(x, t)$ be the solution of

$$\begin{aligned} &h_t - \Delta_x h = 0, && x \in H,\ t>0 \\ &h(x,0)=0, && x\in H \\ &\frac{\partial h}{\partial n(x)}(x,t)\Big|_{x\in\partial H} = \frac{\partial v_N^*}{\partial n(x)}(x,t)\Big|_{x\in\partial B}, && t>0, \end{aligned} \tag{3.17_N}$$

where H is the supporting halfspace at x_0 with $B_1 \subset \Omega \subset H$, then

$$h_N^*(x, t) \leqslant h_N(x, t) \qquad \text{on} \quad B$$

by [4, p. 475, Theorem 2].

Let z_N be the solution of (3.17_N) in the rescaled (y, τ) variables given by (3.15). Then by Theorem 3.3 and its corollary

$$\frac{\partial z_N}{\partial n(y)}(y, \tau) > c \qquad \text{for all} \quad y \in \partial H,\ \tau > \tau_c. \tag{3.18}$$

We now can obtain a lower bound on the growth of z_N. Consider

$$\begin{aligned} &z_\tau - \Delta_y z = 0, && y \in H,\ \tau > \tau_c \\ &\frac{\partial z}{\partial n}(y, \tau) = c, && y \in \partial H,\ \tau > \tau_c \\ &z(y, \tau_c) = 0, && y \in H. \end{aligned} \tag{3.19}$$

THEOREM 3.4. *If*

$$S(y, \tau) = c(\tau - \tau_c)^{1/2}\, g\left(\frac{y}{(\tau - t_c)^{1/2}}\right) \tag{3.20}$$

is the self-similar solution of (3.19) *for* $\tau > \tau_c$, *then, for* $2 \leqslant L < \infty$, *there exists* $c_2 > 0$ *such that*

$$z_N(y, \tau) \geqslant c_2 \tau^{1/2} \qquad \textit{as} \quad \tau \to \infty \tag{3.21}$$

for all $y = (y_1, ..., y_n) \in H$ *with* $0 \leqslant y_1 \leqslant L$.

Proof. Choose c_2 so that $0 < c_2 < 2c/\sqrt{\pi}$. Then since $s(y, \tau)$ is the self-similar solution of (3.19) and $z_N(y, \tau)$ satisfies (3.18), we have for all y with $0 \leqslant y \leqslant L$ that

$$z_N(y, \tau) \geqslant c_2 \tau^{1/2} \qquad \text{as} \quad \tau \to \infty.$$

COROLLARY. *The solution* $h_N^*(x, t)$ *of* (3.17_N) *satisfies*

$$h_N^*(x, t) \geqslant c_2 (te^N)^{1/2} \tag{3.22}$$

for $te^N > K$ *for some* K *sufficiently large and all* $x \in \Omega$ *with* $\operatorname{dist}(x, \partial\Omega) \leqslant Le^{-N/2}$.

For $\mathscr{F} > K$, consider

$$\begin{aligned} &w'_\tau - \Delta_y w' = (1 - w')\, e^{z_N(y,\tau)} \\ &w'(y, \tau) = 0, \qquad y \in \partial B,\ B \subset B_1 \subset \Omega \\ &w'(y, \mathscr{F}) = 0, \qquad y \in B, \end{aligned} \tag{3.23}$$

where, in x,

$$B = B_{e^{-N/2 - c_2 \mathscr{F}^{1/2}/2}} \subset B_{e^{-N/2}}.$$

Since (3.21) holds for $0 \leqslant y_1 \leqslant L$, $\tau \geqslant \mathscr{F}$, the solution $w'(y, \tau)$ of (3.23) satisfies the differential inequality

$$w'_\tau - \Delta_y w' \geqslant (1 - w')\, e^{c_2 \mathscr{F}^{1/2}} \tag{3.24}$$

with the initial-boundary conditions of (3.23).

We again rescale letting

$$s = e^{c_2 \mathscr{F}^{1/2}} (\tau - \mathscr{F}), \qquad \xi = e^{c_2 \mathscr{F}^{1/2}/2} y \tag{3.25}$$

then the solution $\bar{w}(\xi, s)$ of

$$\begin{aligned} &\bar{w}_s - \Delta_\xi \bar{w} = 1 - \bar{w} \\ &\bar{w}(\xi, 0) = 0, \qquad \xi \in B \\ &\bar{w}(\xi, s) = 0, \qquad \xi \in \partial B,\ s \geqslant 0 \end{aligned} \tag{3.26}$$

satisfies, analogous to Theorems 3.1 and 3.3, the following.

For some given $c_3 \in (0, 1)$, there exists $\tau_{c_3} > 0$ such that

$$-\bar{w}_{n(\xi)}(\xi, s) > c_3 \qquad \text{for} \quad \xi \in \partial B,\ s \geqslant \tau_{c_3} \tag{3.27}$$

and hence

$$-w'_{n(y)}(y, \tau) > c_3 e^{c_2 \mathscr{F}^{1/2}/2} \qquad \text{for} \quad \tau \geqslant \tau_{c_3} e^{-c_2 \mathscr{F}^{1/2}} + \mathscr{F} \tag{3.28}$$

since w' is an upper solution for (3.26) by (3.24). But since $z_N = h_N^* \leqslant h_N$, $v_N \geqslant w'$, and $-\partial v_N / \partial n \geqslant -\partial w' / \partial n$,

$$\frac{\partial h_N}{\partial n(x)} \geqslant -\frac{\partial w'}{\partial n(x)}. \tag{3.29}$$

From (3.28) and (3.29), for $\tau = 2\mathscr{F}$, we thus have

$$\frac{\partial h}{\partial n(x)}(x, t) \geqslant c_3 e^{N/2 + c_4 \tau^{1/2}} \geqslant c_3 e^{N/2 + c_4 e^{N/2} t^{1/2}}, \tag{3.30}$$

where $t = e^{-N} \tau$ and $x \in \partial H$ provided $2\mathscr{F} \geqslant \tau_{c_3} e^{-c_2 \mathscr{F}^{1/2}} + \mathscr{F}$ and $\mathscr{F} > K$.

The lower bound estimate (3.30) can now be used to obtain an improved upper bound on t_N. From (3.30) we have

$$h(x,t) \geqslant \int_{2K'e^{-N}}^{t} k(x, t-s)\, c_3 e^{N/2 + c_4 e^{N/2} s^{1/2}}\, ds \tag{3.31}$$

for $t \geqslant 2K'e^{-N}$, where

$$K' = \max[K, \text{ solution of } \mathscr{F} = \tau_{c_3} e^{-c_2 \mathscr{F}^{1/2}}].$$

Thus, letting $D = \operatorname{diam} \Omega$

$$\begin{aligned} h(x,t) &\geqslant c_5 \int_{2K'e^{-N}}^{t} (t-s)^{-1/2} \\ &\quad \times \exp\left(\frac{-D^2}{4(t-s)}\right) e^{N/2 + c_4 e^{N/2} s^{1/2}}\, ds \\ &= c_5 \int_{2K'e^{-N}/t}^{1} (t-ts)^{-1/2} \\ &\quad \times \exp\left[\frac{-D^2}{4t(1-s)} + \frac{N}{2} + c_4 e^{N/2} t^{1/2} s^{1/2}\right] ds \end{aligned} \tag{3.32}$$

by the change of variable $s \to ts$

$$\begin{aligned} &= t^{1/2} c_5 e^{N/2} \int_{2K'e^{-N}/t}^{1} (1-s)^{-1/2} \\ &\quad \times \exp\left[c_4 e^{N/2} t^{1/2} s^{1/2} - \frac{D^2}{4t(1-s)}\right] ds. \end{aligned}$$

Take $t = Ae^{-N/3}$ for some positive A. For N large enough, we clearly have $t > 2K'e^{-N}$. Then

$$\begin{aligned} h(x,t) &\geqslant A^{1/2} e^{-N/6} c_5 e^{N/2} \int_{2K'e^{-N}/Ae^{-N/3}}^{1} (1-s)^{-1/2} \\ &\quad \times \exp\left[c_4 e^{N/3} A^{1/3} s^{1/2} - \frac{D^2}{4Ae^{-N/3}(1-s)}\right] ds. \end{aligned} \tag{3.33}$$

The dominant contribution to the integral in (3.33) is from s near S, where the argument of the exponential $I = e^{N/3}[c_4 A^{1/2} s^{1/2} - D^2/4A(1-s)]$ takes on its maximal value. I is maximal if

$$\frac{1}{2} c_4 A^{1/2} S^{-1/2} = \frac{D^2}{4A(1-S)^2}$$

so A and S are related by

$$2c_4 A^{3/2} = \frac{D^2 S^{1/2}}{(1-S)^2}$$

or

$$A = (2c_4)^{-2/3} D^{4/3} \frac{S^{1/3}}{(1-S)^{4/3}}, \qquad 0 < S < 1. \tag{3.34}$$

In this case,

$$\begin{aligned} I &= \left[c_4 A^{1/2} S^{1/2} - \frac{D^2}{4A(1-S)} \right] e^{N/3} \\ &= 2^{-4/3} e^{N/3} D^{2/3} c_4^{2/3} S^{-1/3} (1-S)^{-2/3} (3S-1). \end{aligned}$$

Taking A to be given by (3.34) for some $S \in (\frac{1}{3}, 1)$, we see that the argument of the integral near $s = S$ is itself exponentially large of order $O(e^{N/3})$ and $h(x, t) \geqslant 1$ for $t \geqslant Ae^{-N/3}$.

In particular for N sufficiently large

$$h(x, t) \geqslant 1 \qquad \text{for} \quad t \geqslant \tau_N$$

with $\tau_N = Ae^{-N/3}$ and hence $h(x, t_N) \geqslant 1$ for all $x \in \Omega$, where

$$t_N \leqslant \tau_N = Ae^{-N/3}.$$

But $t_N \leqslant Ae^{-N/3}$ implies $T < \infty$ which is a contradiction.

4. Robin–Dirichlet Boundary Conditions

In this section, we prove blowup in finite time for (1.1) with Robin–Dirichlet boundary conditions provided $f(u) = e^u$ and $\Omega = B$ is a ball in $\mathbb{R}^n$.

More precisely, consider

$$\begin{aligned} &u_t - \Delta u = (1-v)\, e^u, && x \in B,\ t > 0 \\ &v_t - \Delta v = (1-v)\, e^u, && \\ &u_n + \mu u = 0, \qquad v = 0, && x \in \partial B,\ t > 0,\ \mu > 0 \\ &u(x, 0) = u_0(x) \geqslant 0, \qquad 0 \leqslant v(x, 0) = v_0(x) \leqslant 1, && x \in B, \end{aligned} \tag{4.1}$$

where $B = B_R \subset \mathbb{R}^n$ is a ball of sufficiently large radius R that no steady state solutions exist, $u_0(x)$ and $v_0(x)$ are assumed to be radially symmetric on B,

$$\Delta u_0(x) + (1 - v_0(x))\, e^{u_0(x)} \geqslant 0, \qquad \Delta(u_0 - v_0) \geqslant 0, \qquad v_0(x) \in C(\bar{B}),$$

and ∇v_0 is bounded near ∂B.

The proof of blowup is quite similar to argument used in Section 3 for the Neumann–Dirichlet boundary conditions.

We again consider an equivalent problem by setting $h = u - v$. Then (4.1) is equivalent to

$$\begin{aligned} &u_t - \Delta u = (1 + h - u)\, e^u \\ &h_t - \Delta h = 0 \\ &u(x, 0) = u_0(x), \qquad h(x, 0) = u_0(x) - v_0(x) \\ &u_n + \mu u = 0, \qquad u = h, \qquad x \in \partial B. \end{aligned} \tag{4.2}$$

By the assumptions on $u_0(x)$ and $v_0(x)$ we have

$$u_t(x, t) \geqslant 0, \qquad h_t(x, t) \geqslant 0, \qquad \text{on} \quad B \times [0, T). \tag{4.3}$$

Assume that $T = +\infty$, then $h(x, t) \to \infty$ for all $x \in B$. Otherwise, $h(R, t)$ and consequently $h(r, t)$ is bounded and h, u necessarily have to converge to a nonexistent steady state.

For each $N \in \mathbb{N}$, define

$$T_N \equiv \inf\{t > 0 \colon h(x, s) \geqslant N \text{ for } x \in B,\ t < s\}, \tag{4.4}$$

then

$$T_N < T_{N+1}, \qquad h(x, T_N) \geqslant N \qquad \text{for all} \quad x \in B,$$

and $T_1 + \sum_{N=1}^{\infty} (T_{N+1} - T_N) = \infty$.

As in Section 3, we again proceed to construct a sequence $\{\tau_N\}$ with $\tau_N > T_{N+1} - T_N$ such that $\sum \tau_N < \infty$ which contradicts the assumption of infinite time blowup.

For each $N \in \mathbb{N}$, let (v_N, h_N) be the unique solution of

$$\begin{aligned} &v_t - \Delta v = (1 - v)\, e^{v + h + N}, && x \in B,\ t > 0 \\ &h_t - \Delta h = 0, && \\ &v(x, 0) = 0, \qquad h(x, 0) = 0, && x \in B \\ &v = 0, \qquad (v + h)_N + \mu(v + h) = 0, && x \in \partial B,\ t > 0. \end{aligned} \tag{4.5$_N$}$$

then as, in Section 3,

$$h_N(x, t) \leqslant h_N(x, t) + N \leqslant h(x, t + T_N), \qquad x \in B,\ t > 0. \tag{4.6}$$

Set $t_N = \inf\{t > 0\colon h_N(x, s) \geqslant 1,\ x \in B,\ s \geqslant t\}$, then $h_N(x, t_N) \geqslant 1$ for all $x \in B$ and $T_{N+1} \leqslant T_N + t_N$.

For arbitrary $x_0 \in \partial B$ and $N \in \mathbb{N}$ sufficiently large so that $e^{-N/2} < R$, consider the interior ball $B_{e^{-N/2}}$ tangent to B_R at x_0. Let $w_N(x, t)$ be the solution of

$$\begin{aligned} &w_t - \Delta_x w = (1 - w)\, e^N, \qquad x \in B_{e^{-N/2}},\ t > 0 \\ &w(x, 0) = w_0(x) \\ &w(x, t) = 0. \end{aligned} \tag{4.7$_N$}$$

Rescale by letting

$$x = e^{-N/2} y, \qquad t = e^{-N}\tau \tag{4.8}$$

then $w_N(y, \tau)$ is the solution of

$$\begin{aligned} &w_\tau - \Delta_y w = (1 - w), && y \in B,\ \tau > 0 \\ &w(y, 0) = 0, && y \in B_1 \\ &w(y, \tau) = 0, && y \in \partial B_1. \end{aligned} \tag{4.9$_N$}$$

Then again, as in Section 3, we have that for any $c \in (0, 1)$, there exists $\tau_c > 0$ such that

$$-\frac{\partial w_N(y, \tau)}{\partial n(y)} > c \qquad \text{for} \quad y \in \partial B_1,\ \tau > \tau_c, \tag{4.10}$$

and hence

$$-\left.\frac{\partial v_N(x, t)}{\partial n(x)}\right|_{x \in \partial B_R} > c e^{N/2} \qquad \text{for} \quad t > \mathscr{T}_c \equiv \tau_c e^N. \tag{4.11}$$

Since $h_N(x, t)$ satisfies

$$\begin{aligned} &h_t - \Delta h = 0 \\ &h(x, 0) = 0 \\ &h_n(x, t) + \mu h(x, t)\,|_{x \in \partial B} = -\frac{\partial v_N}{\partial n(x)}(x, t)\,|_{x \in \partial B} \end{aligned}$$

by (4.5_N), then by (4.11)

$$\frac{\partial h_N(x, t)}{\partial n(x)} + \mu h_N(x, t)|_{x \in \partial B} > ce^{N/2} \qquad \text{for} \quad t > \mathscr{T}_c. \tag{4.12}$$

Let $\tilde{h}_N$ be the solution of

$$\begin{aligned} &h_t - \Delta h = 0 \\ &h(x, \mathscr{T}_c) = 0, && x \in B \\ &h_n + \mu h = ce^{N/2}, && x \in \partial B \end{aligned} \tag{4.13_N}$$

then $h_N(x, t) \geqslant \tilde{h}_N(x, t) \geqslant 0$ for all $x \in B$, $t > \mathscr{T}_c$.

Since $\bar{H}(x, t) = (e^{-N/2}/c)\, \tilde{h}_N(x, t)$ satisfies

$$\begin{aligned} &h_t - \Delta h = 0, && x \in B \\ &h(x, \mathscr{T}_c) = 0, && x \in B \\ &h_n(x, t) + \mu h = 1, && x \in B,\ t > \mathscr{T}_c \end{aligned} \tag{4.14_N}$$

there exists $\mathscr{T}_d > \mathscr{T}_c$ such that

$$\bar{H}(x, t) \leqslant \alpha/c$$

for some $\alpha < c/\mu$ and $t \in (\mathscr{T}_c, \mathscr{T}_d)$. This implies

$$\tilde{h}_N(x, t) < \alpha e^{N/2}, \qquad x \in \partial B,\ t \in (\mathscr{T}_c, \mathscr{T}_d) \tag{4.15}$$

and hence, from (4.12) and (4.15)

$$\frac{\partial \tilde{h}_N(x, t)}{\partial n(x)} > (c - \alpha\mu)\, e^{N/2} \tag{4.16}$$

for $x \in \partial B$, $t \in (\mathscr{T}_c, \mathscr{T}_d)$.

Let $\bar{h}_N(x, t)$ be the solution of

$$\begin{aligned} &h_t - \Delta h = 0, && x \in H,\ t > 0 \\ &h(x, \mathscr{T}_c) = 0, && x \in H \\ &\frac{\partial h}{\partial n(x)}(x, t) = (c - \alpha\mu)\, e^{N/2}, && x \in \partial H,\ t > 0, \end{aligned} \tag{4.17_N}$$

where H is the supporting half-space to B at x_0, then

$$\tilde{h}_N(x, t) \geqslant \bar{h}_N(x, t) \tag{4.18}$$

for all $x \in B$, $t \in (\mathscr{T}_c, \mathscr{T}_d)$ by [4, p. 475, Theorem 2].

We now can proceed as in Section 3 to show that for any $L \in (2, \infty)$

$$h_N(x, t) \geq \tilde{h}_N(x, t) \geq \bar{h}_N(x, t) \geq c_2(te^N)^{1/2} \tag{4.19}$$

for all $x \in B$ with $\operatorname{dist}(x, \partial B) < Le^{-N/2}$ and $Ke^{-N} < t < \mathscr{F}_d$ for $K > 0$, $N \in \mathbb{N}$ sufficiently large. By continuing the argument as given in Section 3, we have

$$-\frac{\partial v_N}{\partial n(x)}(x, t) \geq c_3 e^{N/2 + c_2 \mathscr{F}^{1/2}/2} \tag{4.20}$$

for $\mathscr{F} < K$, $x \in \partial B$, and $\mathscr{F}_d > t > \mathscr{F} e^{-N} + \mathscr{F}_{c_3} e^{-N - c_2 \mathscr{F}^{1/2}}$. Then

$$\frac{\partial h_N(x, t)}{\partial n(x)} + \mu h_N(x, t) \geq c_3 e^{N/2 + c_2 \mathscr{F}^{1/2}/2} \tag{4.21}$$

for $x \in \partial B$ and $\mathscr{F} e^{-N} + \mathscr{F}_{c_3} e^{-N + c_2 \mathscr{F}^{1/2}} < t < \mathscr{F}_d$. Again as in Section 3, using similar arguments which took us from (4.12) to (4.20), we can conclude from (4.21) that

$$h_N(x, t) \geq c_5 e^{N/2 + c_2 \mathscr{F}^{1/2}/2} t^{1/2} \tag{4.22}$$

for $x \in \partial B$, $2\mathscr{F} e^{-N} \leq t \leq \mathscr{F}_e$, where

$$\mathscr{F} \geq K' = \max[K, \text{root of } \mathscr{F} = \mathscr{F}_{c_3} e^{-c_2 \mathscr{F}^{1/2}}].$$

By taking $t = 2\mathscr{F} e^{-N}$, we then have

$$h_N(x, t) \geq c_5 e^{N/2 + c_4 e^{N/2} t^{1/2}} \tag{4.23}$$

for $x \in \partial B$, $2K' e^{-N} \leq t < \mathscr{F}_e$.

The proof now proceeds as in Section 3 to conclude that $T = T_1 + \sum (T_{N+1} - T_N) < \infty$ which contradicts our assumption that $T = \infty$. We conclude finite time blowup.

Remark. The proof could be carried out for a convex domain with an interior ball condition as in Section 3.

5. Discussion

We now discuss informally the possible solution behavior for solutions of (1.1) for each of the nine possible boundary condition combinations $(\mathrm{I}_1, \ldots, \mathrm{III}_3)$.

Denote, for this section, (1.1) by (5.1) and again, setting $h = u - v$, consider

$$\begin{aligned}
&u_t - \Delta u = (1 + h - u) f(u) \\
&h_t - \Delta h = 0 \\
&u(x, 0) = u_0(x), \qquad h(x, 0) = u_0(x) - v_0(x) \geqslant -1 \\
&u_n + \mu u = 0, \qquad h_n + vh = (v - \mu) u, \qquad v, \mu < \infty \\
&h = u, \qquad v = \infty \\
&h_n + vh = u_n, \qquad v < \infty = \mu.
\end{aligned} \tag{5.2}$$

Note that for $\mu \leqslant v$, (5.2) is quasi-monotone and hence the standard comparison theorems hold.

We now proceed to discuss each boundary value combination to complete our analysis of Fig. 1.

(I_1) (Dirichlet–Dirichlet) $\mu = v = \infty$.

Steady-state (SS) solutions exist for any f, Ω and the standard comparison theorems are valid.

For (5.2), $(\alpha_1, \alpha_2) = (0, 0)$ is a lower solution and $(\beta_1, \beta_2) = (U, H)$, where (U, H) is a steady-state solution for (5.2) with boundary condition for u replaced by $U = A$ $(\mu = \infty)$ and A sufficiently large to ensure $U(x) > A > \max[\sup u_0(x), \sup h_0(x)]$, is an upper solution.

Thus, the solution of (5.2) and hence of (1.1) exists globally.

If (u_1, h_1) is the minimal steady-state solution and (u_2, h_2) is the maximal steady-state solution, then $(u_0, h_0) < (u_1, h_1)$ implies $(u, h) \to (u_1, h_1)$ and $(u_0, h_0) > (u_2, h_2)$ implies $(u, h) \to (u_2, h_2)$.

If the steady-state solution is unique (Ω (or f) large enough or small enough guarantees this), then (5.2) and hence (1.1) is globally asymptotically stable (GAS).

(I_2) (Dirichlet–Robin) $\mu = \infty > v > 0$.

Steady-state solutions exist for any f, Ω.

For (5.2), take $K > \max(\sup u_0, 1, 1 + \sup h_0)$. Then $(\beta_1, \beta_2) = (K, K - 1)$ is an upper solution for (5.2). Thus, (u, h) is bounded above and there exists C such that $-u_n < C$. Take $L \geqslant \max(-\inf h_0(x), 1, C)$. Then $(1 - L, -L) = (\alpha_1, \alpha_2)$ is a lower solution for (5.2). Thus, we have global existence for (5.2) and hence for (1.1).

We conjecture global asymptotic stability for f (or Ω) very small.

(I_3) (Dirichlet–Neumann) $\mu = \infty$, $v = 0$.

The steady-state solution is (0, 1) for any f, Ω.

$$u_t - \Delta u = (1-v)\, f(u), \qquad u_n = 0$$
$$v_t - \Delta v = (1-v)\, f(u), \qquad v_n \leqslant 0$$

implies $(K, 1)$ is an upper solution for some $K > 0$ and $u \leqslant K + v \leqslant K+1$. Then $f > 0$ implies $v \to 1$ as $t \to \infty$ and $u \to 0$ as $t \to \infty$. Thus, the trivial steady-state solution (0, 1) is globally asymptotically stable.

(II_1) (Robin–Dirichlet) $0 < \mu < \infty = \nu$.

(a) For f (or Ω) large enough, then no steady-state solution exists and solutions to (1.1) blow up.

(b) (i) For f (or Ω) small, then steady-state solutions exist. If the initial data is too large so that $(u_t)_0 \geqslant 0$, $(h_t)_0 \geqslant 0$ with $u >$ any steady-state solution, then the solution to (1.1) blows up.

(ii) For f (or Ω) small, then steady-state solutions exist. If the initial data is small, e.g., $(u_0, h_0) <$ minimal steady-state solution, then monotonicity implies $(u, h) \to$ minimal steady-state and the minimal steady-state solution is asymptotically stable.

Remark. We have only proved blowup for $f(u) = ae^{bu}$ and $\Omega = B$. The proof may be adapted for f growing faster. We conjecture that blowup occurs for (a) and (b_i) whenever $f(u) > O(u^2)$ for u large.

(II_2) (Robin–Robin) $0 < \mu$, $\nu < \infty$.

There exist steady-state solutions for any f or Ω.

(a) $\mu \leqslant \nu$ implies monotonicity and same conclusions as in (I_1).

(b) $\mu > \nu$ implies same conclusions as in (I_2).

(II_3) (Robin–Neumann) $\nu = 0 < \mu < \infty$.

Same as for (I_3).

(III_1) (Neumann–Dirichlet) $\mu = 0$, $\nu = \infty$.

There exists no steady-state solution for any f or Ω—as in (II_1)(a).

Remark. Again the proofs we have given only apply for exponentially (or faster) growing f when Ω is convex and for radially symmetric problems for more general f. We conjecture however that blowup occurs for quite general problems.

(III_2) (Neumann–Robin) $\mu = 0 < \nu < \infty$.

There exists no steady-state solution for any f or Ω.

Let $w(x)$ be the solution of torsion problem

$$-\Delta w = 1$$
$$w = 0.$$

Let $\varphi(x) = kw(x)$, where k is such that $-kw_n(x) \geqslant v$. Choose $b \geqslant \sup(h_0 + \varphi)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right)(at + b - \varphi) = a - k \geqslant 0 \qquad \text{if} \quad a \geqslant k$$

and, noting that $h_n + v_n = h_n - \nu v = 0$, $v \leqslant 1$,

$$\frac{\partial}{\partial n}(at + b - \varphi) - \nu v \geqslant -kw_n - \nu \geqslant 0.$$

Thus, $at + b - \varphi$ is an upper solution for h and $h \leqslant at + b - \varphi$.

Therefore, there is no blowup and we have global existence.

If u is bounded, then v is bounded away from 1. This implies u grows linearly which is a contradiction.

Therefore, u is unbounded as $t \to \infty$.

(III_3) (Neumann–Neumann) $\mu = \nu = 0$.

There exists a one-parameter family of steady-state solutions: $h = C$, $v = 1$, $u = 1 + C$.

$$h_t - \Delta h = 0, \qquad h_n = 0 \qquad \text{implies} \quad h \to C = \bar{h}_0 \text{ as } t \to \infty,$$

where the overbar denotes the average over Ω.

$$v_t - \Delta v = (1 - v)\, f(v + h), \qquad v_n = 0 \qquad \text{implies} \quad v \to 1 \text{ as } t \to \infty.$$

We therefore have global existence with $(u, v) \to (1 + \bar{u}_0 - \bar{v}_0, 1)$ as $t \to \infty$.

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